

# INSTITUT DES HAUTES ÉTUDES

POUR LE DÉVELOPPEMENT DE LA CULTURE, DE LA SCIENCE ET DE LA TECHNOLOGIE EN BULGARIE

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## Concours Général de Mathématiques

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### Solutions

**Exercise 1.** a) Denote by  $O$  the intersection of  $B_1B_2$  and  $C_1C_2$ , and by  $A_3$  the intersection of  $A_1O$  and the circle. Since the triangles  $A_1B_1O$  and  $B_2A_3O$  are similar, we have  $A_1B_1 \cdot B_2O = A_3B_2 \cdot A_1O$ . The same way  $C_1A_3 \cdot A_1O = C_2A_1 \cdot C_1O$  and  $B_2C_2 \cdot C_1O = B_1C_1 \cdot B_2O$ . Multiplying these we get

$$A_1B_1 \cdot C_1A_3 \cdot B_2C_2 = B_1C_1 \cdot A_3B_2 \cdot C_2A_1.$$

Therefore  $A_1B_1 \cdot C_1A_2 \cdot B_2C_2 = B_1C_1 \cdot A_2B_2 \cdot C_2A_1 \iff C_1A_2 \cdot A_3B_2 = A_2B_2 \cdot C_1A_3 \iff C_1A_2B_2$  and  $C_1A_3B_2$  are similar, hence equal  $\iff A_2 = A_3$ .

b) By several applications of the sinus theorem we get :

$$\frac{B_1A_2^2}{A_2C_1^2} = \frac{\sin(CAM)}{\sin(BAM)}, \quad \frac{C_1B_2^2}{B_2A_1^2} = \frac{\sin(ABM)}{\sin(CBM)}, \quad \frac{A_1C_2^2}{C_2B_1^2} = \frac{\sin(BCM)}{\sin(ACM)}.$$

Then b) follows from a) and from :

$$\frac{B_1A_2^2 \cdot C_1B_2^2 \cdot A_1C_2^2}{A_2C_1^2 \cdot B_2A_1^2 \cdot C_2B_1^2} = \frac{\sin(CAM) \sin(ABM) \sin(BCM)}{\sin(ACM) \sin(CBM) \sin(BAM)} = \frac{AM \cdot BM \cdot CM}{BM \cdot CM \cdot AM} = 1.$$

**Exercise 2.** It is easy to construct special boards for  $n = 2$  and  $n = 4$ .

Assume that  $n \geq 3$  is an odd number, and take an  $n \times n$  board containing the numbers  $1, 2, \dots, 2n - 1$ . Let  $i$  ( $1 \leq i \leq 2n - 1$ ) be a number that is not on the principal diagonal (such an  $i$  exist since the principal diagonal contains  $n$  cells). Then the number  $i$  is contained in the union of the  $k$ -th row and the  $k$ -th column for an even number of  $k$ 's. Since  $k$  takes an odd number of values ( $1 \leq k \leq n$ ), the board cannot be special.

**Exercise 3.** Denote by  $\alpha$ ,  $\beta$  and  $\gamma$  the angles of the triangle. Then

$$\frac{BC}{r} = \cotg\left(\frac{\beta}{2}\right) + \cotg\left(\frac{\beta}{2}\right).$$

a) We have  $\alpha \leq \beta \leq \gamma < \frac{\pi}{2}$ . Since we look for the maximal possible value for  $\frac{BC}{r}$  and since the function  $\cotg$  is decreasing, we may assume that  $\alpha = \beta$ . We look for the maximum of  $f(\beta) = \cotg\left(\frac{\beta}{2}\right) + \cotg(\pi - \beta)$ , when  $\beta \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ . Put  $t = \tan\left(\frac{\beta}{2}\right) \in [\sqrt{2} - 1, \frac{1}{\sqrt{3}}]$ . Then

$$f(\beta) = F(t) = \frac{1+t^2}{t-t^3} \text{ and } F'(t) = \frac{t^4+4t^2-1}{(t-t^3)^2}.$$

It is easy to deduce that  $F(t)$  is decreasing on  $[\sqrt{2} - 1, \sqrt{\sqrt{5} - 2}]$  and increasing on  $[\sqrt{\sqrt{5} - 2}, \frac{1}{\sqrt{3}}]$ .

The maximum is  $2\sqrt{3} = F\left(\frac{1}{\sqrt{3}}\right) > F(\sqrt{2} - 1)$ . Therefore the maximal possible value for  $\frac{BC}{r}$  is  $2\sqrt{3}$  and is reached when the triangle is equilateral.

b) By the above study the minimal possible value for  $\frac{BC}{r}$  is equal to  $F(\sqrt{\sqrt{5} - 2}) = \frac{\sqrt{5}-1}{(3-\sqrt{5})\sqrt{\sqrt{5}-2}}$ .

**Exercise 4.** The glutton finishes always by eating all the candies after less than  $2(n+k)$  steps. It is obvious that  $P(0, k) = 0$ ,  $P(n, 0) = 1$  and  $P(1, 1) = 1/2$ . We will prove by induction on  $n+k$  that  $P(n, k) = 1/2$ , if  $n \geq 1$  and  $k \geq 1$ . Assume that for all  $i \geq 1$ ,  $j \geq 1$ ,  $i+j < n+k$ , we have  $P(i, j) = 1/2$ . By counting what happens between the beginning and the next time that glutton arrives in case (i), we obtain :

$$\begin{aligned}
P(n, k) &= \frac{k}{n+k} \left( \frac{n}{n+k-1} P(n, k-1) + \frac{(k-1)n}{(n+k-1)(n+k-2)} P(n, k-2) + \dots \right. \\
&\quad \left. \dots + \frac{(k-1)(k-2)\dots 1 \cdot n}{(n+k-1)(n+k-2)\dots(n+1)n} P(n, 0) \right) + \\
&\quad + \frac{n}{n+k} \left( \frac{k}{n+k-1} P(n-1, k) + \frac{(n-1)k}{(n+k-1)(n+k-2)} P(n-2, k) + \dots \right. \\
&\quad \left. \dots + \frac{(n-1)(n-2)\dots 1 \cdot k}{(n+k-1)(n+k-2)\dots(k+1)k} P(0, k) \right) = \\
&= \frac{n}{2(n+k)} \cdot \frac{k}{n+k} \left( 1 + \frac{k-1}{n+k-2} \left( 1 + \dots + \frac{2}{n+1} \left( 1 + \frac{1}{n} \right) \dots \right) \right) + \frac{k!n!}{2(k+n)!} + \\
&+ \frac{k}{2(n+k)} \cdot \frac{n}{n+k} \left( 1 + \frac{n-1}{n+k-2} \left( 1 + \dots + \frac{2}{k+1} \left( 1 + \frac{1}{k} \right) \dots \right) \right) - \frac{n!k!}{2(k+n)!} = \\
&= \frac{n}{2(n+k)} \cdot \frac{k}{n} + \frac{k}{2(n+k)} \cdot \frac{n}{k} = \frac{1}{2}
\end{aligned}$$

END